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CURVY SLICING PROVES THAT TRIPLE JUNCTIONS LOCALLY MINIMIZE AREA

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Abstract

In soap films three minimal surfaces meet at 120-degree angles. We use a novel curvy slicing argument to prove that small pieces minimize area for given boundary. The argument applies in general dimension and codimension.

1. Introduction

The Belgian physicist J.A.F. Plateau [18] observed and recorded two types of singularities in soap films:

(1) three minimal surfaces meeting smoothly at angles of 120 degrees along a curve,

(2) four such curves meeting smoothly at angles of about 109 degrees at a point.

J. Taylor [20] proved that certain locally area-minimizing surfaces ($(\mathbf{M}, 0, \delta)$-minimal sets) have precisely these two types of singularities. For the first type, our Theorem 5.1 proves conversely that any such configuration is locally area minimizing in a stronger sense (among separators of regions). We extend our result to general dimension and codimension. Even for a regular piece of minimal surface, such area minimization is by no means obvious, and we begin with a new simple proof of this case (Theorem 2.1).

Whether "tetrahedral" singularities of type (2) locally minimize area remains an open question, although paired calibrations [*LM*] prove the tetrahedral cone itself minimizing in all dimensions. A more general conjecture states that a stationary singular surface minimizes area in

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a neighborhood of any point where it has a strictly area-minimizing tangent cone. Some such results for minimal surfaces with isolated singularities were proved by Hardt and Simon for one notion of "strictly minimizing" [7, §3] and by Lawlor [12, Thm. 6.3.1] for a related notion [12, 6.1.1].

Curvy slicing. Before discussing our new curvy slicing, we recall the proof by standard slicing that a piece S of three vertical halfplanes meeting along the z-axis at angles of 120 degrees is area minimizing. Each slice by a horizontal disc is a "Y" consisting of three line segments meeting at angles of 120 degrees. Each Y is length minimizing. Since S is perpendicular to the horizontal slicing discs, its area equals the integral of the lengths of the slices, and hence is as small as possible.

In the general case, when S is a piece of three minimal surfaces meeting at angles of 120 degrees, we slice by curvy discs orthogonal to S. Because the distance between such discs is variable, the area of S equals the integral of the lengths of the slices in a certain weighted metric, in which the slices consist of geodesics. The crucial estimate is that the metric is $C^{1,1}$ (although not in general C^2). The requisite Lemma (Proposition 4.1) states that a geodesic Y in a $C^{1,1}$ metric is locally length minimizing. For a general $C^{1,\alpha}$ metric for $\alpha < 1$, even a single geodesic can fail to be locally length minimizing.

Organization. To illustrate our methods, Section 2 gives a simple proof by curvy slicing that regular minimal surfaces are locally area minimizing. Sections 3 and 4 provide the needed lemmas on $C^{1,1}$ metrics: normal coordinates and locally minimizing Y's. Section 5 contains our main result, that triple junctions are locally minimizing.

2. Regular minimal surfaces are locally area minimizing

Theorem 2.1 illustrates our slicing methods with a new, simple proof that a regular minimal surface is locally area minimizing in general dimension and codimension, even in competition with unoriented surfaces.

Classical geodesic field theory first proved for $m = 1$ that geodesics are locally minimal and later proved for $m = n - 1$ that minimal hypersurfaces are locally minimizing, but never succeeded for general m, despite extensive work by many mathematicians, including Weyl and Caratheodory (see [6]). Finally Federer [4, 1975] gave an intricate proof for general m in the category of oriented surfaces (rectifiable currents). Morgan

[13, Cor. 3.4, 1984] admitted unoriented competitors in a weak compactness proof using White's spaces of immersions and Allard's regularity theory. Lawlor [12, Thm. 6.2.8, 1991] gave a simpler proof using projections. The following proof based on Lawlor's slicing theory [11] seems to be the best yet.

2.1. Theorem. *A smooth m-dimensional minimal surface S in $\mathbf{R}^n$ is locally area minimizing (even in competition with unoriented surfaces).*

Here "locally area minimizing" means that about every point of S there is a small ball B in which S is area minimizing in competition with any oriented surface S' (rectifiable current) with the same boundary (or such that $\partial S' - \partial S = 2X$ for some rectifiable current X). The extra boundary $2X$ allows one to orient an otherwise nonorientable S'. (We are not admitting the most general nonorientable surfaces of geometric measure theory, which present technical difficulties for slicing.)

Proof. We will slice S into curves by curvy $(n-m+1)$-dimensional surfaces $\{f = c\}$ orthogonal to S, note that the slices are geodesics in an appropriate metric, and conclude that S is locally area minimizing. For convenience we use coordinates $x, y_1, \dots, y_{n-m}, z_1, \dots, z_{m-1}$ on $\mathbf{R}^n$ with S tangent to the x-z plane at the origin and prove S area minimizing in a small ball about $\mathbf{0}$. Let B_1 be a small ball about $\mathbf{0}$ and let C be the slice of $S \cap B_1$ by the y-z plane. In a smaller ball B_2 about $\mathbf{0}$, let P_0 denote nearest point retraction onto C and let P_1 denote nearest point retraction onto S. We confine our attention to B_2. For our slicing function we take the smooth nonsingular map $f = P_0 \circ P_1$. The level sets $\{f = c\}$ slice S orthogonally into curves S_c. By the coarea formula [14, 3.13], we have

$$\operatorname{area} S = \int_C \int_{S_c} 1/J_{m-1}(f|S)dc = \int_C \int_{S_c} 1/J_{m-1} f\, dc.$$

Here J_{m-1} is the Jacobian; for surfaces of dimension $m = 2, J_{m-1} = J_1 = |\nabla f|$. The second equality follows because by construction the level sets of f are perpendicular to S. In general, the restricted Jacobian $J_{m-1}(f|S')$ from the coarea equality is smaller than $J_{m-1}f$, since it is a supremum over a smaller collection. The equation can be restated as

$$\operatorname{area} S = \int_C \operatorname{length}(S_c),$$

where length is measured in the metric $(1/J_{m-1}f)$ times the standard metric on $\{f = c\}$. Since S is stationary, it follows that S_c consists of geodesics in this metric, which are minimal inside some smaller ball B_3.

Now consider any competing surface S' (in B_3) with the same boundary as S (or $\partial S' - \partial S = 2X$). For almost every $c, \partial S'_c$ consists of the same two points as ∂S_c ([14, 4.11], [5, 4.3.1,4.3.8]), possibly plus additional points of multiplicity 2. In any case, S'_c includes a path between the points of ∂S_c and hence length$(S_c) \leq$ length(S'_c). Therefore (in B_3)

$$\text{area}\, S = \int_C \text{length}(S_c) \leq \int_C \text{length}(S'_c) \leq \text{area}\, S'.$$

(The last inequality need not be equality since S' need not be orthogonal to $\{f = c\}$.) We conclude that S is locally area minimizing.

3. Normal coordinates for a $C^{1,1}$ metric

Proposition 3.1 provides normal coordinates for a $C^{1,1}$ metric, as needed in Section 4. The result is standard for a C^2 metric and in general false for a $C^{1,\alpha}$ metric with $\alpha < 1$.

3.1. Proposition. *Consider a ball* $\mathbf{B}(p, \varepsilon_0) \subset \mathbf{R}^n$ *with a* $C^{1,1}$ *metric which is the standard metric at* p. *Then in a smaller neighborhood there are biLipschitz normal coordinates in which for some positive constant* a *distances are bounded above and below by the metric in polar coordinates*

$$dr^2 + (1 \pm ar^2)r^2 d\Theta^2. \tag{1}$$

Here $d\Theta$ is the standard metric on the unit sphere, so that $dr^2 + r^2 d\Theta^2$ is the standard metric on $\mathbf{R}^n$. The size of the smaller neighborhood and the constant a need depend only on a lower bound on the radius ε_0 of the original ball and an upper bound on the $C^{1,1}$ norm of the metric.

Remarks. In biLipschitz coordinates, the Riemannian metric tensor may be undefined on a set of measure 0, but the length of a Lipschitz curve is welldefined via the original coordinates. If the metric turned out to be Hölder continuous in the new coordinates, they would have to be C^1 by a theorem of Calabi and Hartman [3].

The standard theory of differential equations provides a unique C^2 geodesic in every direction from p and therefore a well defined exponential map, but requires a C^2 metric to conclude the exponential map is C^1 and hence a local diffeomorphism. If the metric is merely $C^{1,\alpha}(\alpha < 1)$, geodesics can fail to be locally minimizing or fail to be unique for a given initial direction [9, §5].

P. Hartman [8, Thm. 1.3] proved for example the existence of C^1 normal coordinates for a $C^{1,1}$ metric with Riemannian curvature tensor continuous except at isolated points, a correction of [9, III]. In our application, a uniform version of Hartman's Theorem could replace Proposition 3.1.

Proof. By shrinking ε_0, we may assume the metric stays within a factor of 2 of the standard metric. We consider a smaller ball $\mathbf{B}(p, 9\varepsilon)$, with $4\varepsilon \leq \pi/2K_0$, where K_0 is a bound on the absolute value of the sectional curvature. On $\mathbf{B}(p, 8\varepsilon)$, consider a nice sequence of smoothings converging to the given metric. Then the corresponding sequence of exponential maps f_i satisfy

$$\mathbf{B}(P, 2\varepsilon) \subset f_i(\mathbf{B}(\mathbf{0}, 4\varepsilon)) \subset \mathbf{B}(p, 8\varepsilon). \tag{3}$$

By the Rauch Comparison Theorem, for some constant a, for all $x \in \mathbf{B}(\mathbf{0}, 4\varepsilon)$, in the norm from the metric,

$$(1 - \frac{1}{8}a|x|^2)|\xi| \leq \|Df_i(x)(\xi)\| \leq (1 + \frac{1}{2}a|x|^2)|\xi|. \tag{4}$$

(The factors of 1/8 and 1/2 are for future convenience.) By shrinking ε if necessary, we may assume $a\varepsilon^2 < 1/2$. In particular, f_i is a local diffeomorphism.

We now show that for $0 < r < \varepsilon$, on $\mathbf{B}(\mathbf{0}, r)$, each f_i is a biLipschitz map with

$$(1 - \frac{1}{2}ar^2)|\Delta x| \leq \|\Delta f_i\| \leq (1 + \frac{1}{2}ar^2)|\Delta x|. \tag{5}$$

The second inequality follows immediately from (4). To prove the first, let $x_1, x_2 \in \mathbf{B}(\mathbf{0}, r)$. Let ℓ_1, ℓ_2 denote the images of $\overline{0x_1}, \overline{0x_2}$, and let ℓ_3 denote a shortest path from $f(x_1)$ to $f(x_2)$, which must lie inside $f_i\mathbf{B}(\mathbf{0}, 2r) \subset \mathbf{B}(p, 4\varepsilon)$. Since f_i is a local diffeomorphism we can lift the triangular region $\ell_1\ell_2\ell_3$ to $\mathbf{B}(\mathbf{0}, 2r)$; let $\tilde{\ell}_3$ denote the lift of ℓ_3. By (4),

$$\operatorname{length} \tilde{\ell}_3 \leq (1 - \frac{1}{8}a(2r)^2) \operatorname{length} \ell_3.$$

Therefore $|\Delta x| \leq (1 - \frac{1}{2}ar^2)^{-1}|\Delta f_i|$, proving (5).

On $\mathbf{B}(\mathbf{0}, \varepsilon)$, the f_i converge to a biLipschitz map f satisfying (5). The images of rays from $\mathbf{0}$, as limits of minimal geodesics, are minimal geodesics, so f is the exponential map and $|f^{-1}(q)| = \operatorname{dist}(p, q)$. Consequently whenever f^{-1} is differentiable (almost everywhere), $\nabla r = \partial/\partial r$, and $ds^2 = dr^2 + d\tau^2$, with $d\tau$ tangential. Since f satisfies (5), we have

$$dr^2 + (1 - \frac{1}{2}ar^2)^2 r^2 d\Theta^2 \le ds^2 \le dr^2 + (1 + \frac{1}{4}ar^2)^2 r^2 d\Theta^2$$

and hence

$$dr^2 + (1 - ar^2) r^2 d\Theta^2 \le ds^2 \le dr^2 + (1 + ar^2) r^2 d\Theta^2 \tag{6}$$

almost everywhere.

Now we show that the length of any Lipschitz curve γ in the given $C^{1,1}$ metric is bounded by its lengths in the metrics (6). If not, the reverse inequality can be maintained under smoothing γ in the original coordinates, then taking a small embedded piece, then translating so that (6) holds almost everywhere on γ, to achieve a contradiction.

Finally note that the size of our final small neighborhood and the constant a depended only on the radius ε_0 of the original ball and the $C^{1,1}$ norm of the metric.

4. A geodesic Y in a $C^{1,1}$ metric is locally minimizing

4.1. Proposition. *Let Y be three geodesics meeting at angles of 120 degrees at p in the unit ball with a $C^{1,1}$ metric which is the standard metric at p. Then the portion of the Y inside a small ball about p is uniquely length minimizing. The size of the small ball need depend only on an upper bound on the $C^{1,1}$ norm of the metric.*

Proof. By Proposition 3.1 and scaling, there are biLipschitzian normal coordinates at p in which distances are bounded below by the spherical metric

$$ds^2 = dr^2 + \sin^2 r \, d\Theta^2.$$

Moreover, if the original metric is the identity at p, the size of this coordinate neighborhood need depend only on a lower bound on the size of the original coordinate neighborhood and an upper bound on the $C^{1,1}$ norm of the metric. Since Y is no shorter in this metric, it suffices to prove the result for this metric, i.e., for p the north pole on a relatively small spherical cap. Consider a somewhat smaller piece of the Y, with endpoints p_i, such that the shortest network must lie inside the spherical cap. Distance from p_i is a convex function, strictly convex except in the direction of p_i. Hence the sum of the three distances is a strictly convex function. Since the three geodesic arms of the Y

meet at 120 degrees, p is a critical point, and hence the unique absolute minimum.

Remarks. The hypothesis that the metric be $C^{1,1}$ is necessary. For any $\alpha < 1$, Hartman and Wintner [9, §5] exhibited a $C^{1,\alpha}$ metric on the plane for which the positive y-axis is a geodesic but no initial piece is minimal, namely

$$ds^2 = dx^2 + (1 - |x|^{1+\alpha})dy^2.$$

This metric arises on a C^1 surface of revolution in $\mathbf{R}^3$ [9, §6, Remark p. 138].

5. Triple junctions are locally minimizing

Theorem 5.1 provides the main result of this paper, that three minimal surfaces meeting smoothly at 120 degrees are locally area minimizing. The result holds in general dimension and codimension.

5.1. Theorem. *Three minimal surfaces S_i meeting smoothly (C^4) at 120-degree angles along a curve C in $\mathbf{R}^3$ are locally area minimizing. The same result holds for three $(k+1)$-dimensional surfaces meeting along a k-dimensional surface in $\mathbf{R}^n$.*

"Locally" means that about every point of $S = \cup S_i$ there is a small ball B in which S is area minimizing. The proof actually shows that the size of B need depend only on the C^4 norm of the surfaces and singular "curve" (which may in general be k-dimensional). For the main case of hypersurfaces, as competitors we could allow any set S' which (together with $S \cap B^C$) continues to separate locally the three regions R_i of the complement of S. In particular, S is $(\mathbf{M}, 0, \delta)$-minimal.

To admit higher codimension, where surfaces do not separate space, we give two of the surfaces similar orientations, and we give the third the opposite orientation with multiplicity two (so that there is no boundary along the singular "curve"). Our proof then shows that S minimizes size (area without counting multiplicity, i.e., simply the Hausdorff measure of the underlying rectifiable set). In particular, S is $(\mathbf{M}, 0, \delta)$-minimal. For basic facts about size-minimizing rectifiable currents see [15]. Note that the boundary of almost every 1-dimensional slice will be three points ([14, 4.11], [5, 4.3.1, 4.3.8]). Proposition 4.1 implies that a small geodesic Y, suitably oriented, is size minimizing.

For hypersurfaces, such size minimization implies area minimization among separators of regions. Indeed, any competing separator S' of space into regions R_1, R_2, R_3 can be turned into $\partial R_1 - \partial R_2$ (boundary as currents), which is a competitor to minimize size. Moreover by [5, 4.5.6, 4.5.12], the size of $\partial R_1 - \partial R_2$ is less than or equal to the area (Hausdorff measure) of S'.

An earlier proof [16, Theorem 2.1] using strict calibrations in the larger category of constant-mean-curvature surfaces considered only competitors S' with singular "curve" C' close to the original C in the C^1 norm.

It actually suffices to assume the three surfaces meet with smoothness class $C^{1,\alpha}$ at 120-degree angles, since it follows that they meet real analytically [10, Thm. 5.1].

Proof. We will show how to slice the surfaces orthogonally into Y's which are geodesics in an appropriate $C^{1,1}$ metric, apply Proposition 4.1 to infer the Y's minimizing, and thence deduce the surfaces minimizing.

Since it is known that a small piece of a regular minimal surface is area minimizing, we consider a point p in the singular "curve" C and let $\overline{S}_i$ be a smooth extension of S_i to a nice small neighborhood of p. We confine attention to this small neighborhood.

Let P_i denote nearest point retraction onto $\overline{S}_i$. Let φ give local coordinates on C. Let P_0 denote nearest point retraction onto C. Let (r, θ, z) be cylindrical coordinates on each plane $\{P_0 = c\}$ with $\{\theta = 0, z = 0\}$ tangent to S_1 at C. Let $u(\theta), u'(\theta)$ be nice bump functions equal to 1 near S_2, S_3. Then as $r \to 0$,

$$u = O(1) \quad \text{and} \quad D^m u = O(r^{-m}). \tag{1}$$

The slicing function. Define as our slicing function

$$\begin{aligned} f =& \varphi \circ P_0 \circ P_1 + u(\varphi \circ P_0 \circ P_2 - \varphi \circ P_0 \circ P_1) \\ &+ u'(\varphi \circ P_0 \circ P_3 - \varphi \circ P_0 \circ P_1) \\ =& p_1 + u(p_2 - p_1) + u'(p_3 - p_1), \end{aligned}$$

where P_i is the smooth function $\varphi \circ P_0 \circ P_i$, which gives in local coordinates projection onto C via $\overline{S}_i$.

Clearly f is continuous. To show that f is Lipschitz for example, it suffices to show that Df is bounded for $r > 0$. First note that the $P_0 \circ P_i$ and hence the p_i satisfy

$$p_i - p_j = O(r^2), \quad D(p_i - p_j) = O(r), \quad D^m p_i = O(1). \tag{2}$$

We claim that for unit vectors v, w, with v orthogonal to the level plane of P_0 at a point,

$$D(p_i - p_j)(v) = O(r^2), D^2(p_i - p_j)(v, w) = O(r). \tag{3}$$

The second estimate holds because on C, by (2), D(pi-pj))(w) = 0. The first follows from the second. Hence for $r > 0$, by (1)-(3) we have

$$\begin{aligned} Df =& Dp_1 + Du(p_2 - p_1) + uD(p_2 - p_1) \\ &+ [\text{similar terms in } u', p_3 - p_1, \text{ henceforth omitted}] \\ =& Dp_1 + O(r^{-1})O(r^2) + O(1)O(r) = Dp_1 + O(r) = O(1), \end{aligned} \tag{4}$$

so f is Lipschitz for all r;

$$\begin{aligned} D^2 f =& D^2 p_1 + D^2 u(p_2 - p_1) \\ &+ Du \odot D(p_2 - p_1) + uD^2(p_2 - p_1) \\ =& O(1) + O(r^{-2})O(r^2) + O(r^{-1})O(r) + O(1)O(1) = O(1) \end{aligned} \tag{5}$$

where

$$Du \odot D(p_2 - p_1)(v_1, v_2) = Du(v_1)D(p_2 - p_1)(v_2) + Du(v_2)D(p_2 - p_1)(v_1));$$

so f is $C^{1,1}$. Since Df is nonsingular on C, each level set $\{f = \varphi(c)\}$ is a graph over its tangent plane $\{P_0 = c\}$ at C; let q be the associated $C^{1,1}$ map from $\{P_0 = c\}$ to $\{f = \varphi(c)\}$. Similarly,

$$\begin{aligned} D^3 f =& D^3 p_1 + D^3 u(p_2 - p_1) + D^2 u \odot D(p_2 - p_1) \\ &+ Du \odot D^2(p_2 - p_1) + uD^3(p_2 - p_1) \\ =& O(1) + O(r^{-3})O(r^2) + O(r^{-2})O(r) + O(r^{-1})O(1) \\ &+ O(1)O(1) = O(r^{-1}). \end{aligned} \tag{6}$$

Even though $D^3 f$ in general is $O(r^{-1})$, the essential fact will be that the portions of $D^3 f$ which appear as terms in the second derivatives of $J_k f$ are $O(1)$.

The metric. By the coarea formula [14, 3.13], we have

$$\text{area}\, S = \int_C \int_{S \cap \{f = \varphi(c)\}} \frac{1}{J_k f} = \int_C \text{length}(S_c),$$

where $S = \cup S_i, S_c = S \cap \{f = \varphi(c)\}$ is a slice of S, and length is measured in the metric $(1/(J_k f))$ times the standard metric on $\{f =$

$\varphi(c)\}$. Since S is stationary, it follows that S_c consists of geodesics in this metric (meeting at 120-degree angles). To show that this metric is $C^{1,1}$ in the coordinates of its tangent plane $\{P_0 = c\}$, we show first that the standard metric on $\{f = \varphi(c)\}$ is $C^{1,1}$ and second that the weighting factor $J_k f$ is $C^{1,1}$.

The standard metric. Recall that for a graph of a function $h, ds^2 = dx^2 + dh^2$. In our case, $\{f = \varphi(c)\}$, in the coordinates of $P = \{P_0 = c\}$, has metric

$$ds^2(v) = 1 + |(Df|P^{\perp})^{-1}(Df(v))|^2,$$

where ds is evaluated at x and Df is evaluated at $q(x)$. (For convenience we are assuming $P \cap C = \{0\}$.) To show the metric $C^{1,1}$, it suffices to consider fixed v and bound the second derivatives of the metric on $P - \{0\}$. Since the first two derivatives of f are bounded, we need only consider the third derivative term arising from differentiating the same factor twice, which by (6) is $O(r^{-1})O(r) = O(1)$.

The weighting factor. Therefore we just need to show that the weighting factor $J_k f(q(x))$ is $C^{1,1}$ on P or that $J_k f$ as a function on $\mathbf{R}^n$ is $C^{1,1}$ on $\{f = \varphi(c)\}$. We already know f $C^{1,1}$ and hence $J_k f$ Lipschitz. For convenience, we use the smooth coordinates $(r, \theta, z; \varphi \circ P_0)$ on $\mathbf{R}^n$ so that with orthonormal basis $e_1, \ldots, e_n$,

$$Du(e_i) = 0 \quad \text{for} \quad i > 2 \leq n - k. \tag{7}$$

(Since these coordinates amount to composition with a smooth function, estimates (1)-(6) continue to hold.) Fortunately in the complementary directions, where Du is big, Df is small. Let

$$\xi = e_{n-k+1} \wedge \ldots \wedge e_n, \xi_1, \xi_2, \ldots$$

be the associated orthonormal basis for $\wedge_k \mathbf{R}^n$. Then

$$(J_k f)^2 = |Df(\xi)|^2 + \Sigma |Df(\xi_i)|^2,$$

$$\begin{aligned} D^2(J_k f)^2 =& O(1) + 2(D^2 Df(\xi)) \bullet Df(\xi) + \Sigma(D^2 Df(\xi_i)) \bullet Df(\xi_i) \\ =& O(1) + O(1) D^2 Df(\xi) + O(r^{-1}) O(r), \end{aligned}$$

where $Df(\xi_i) = O(r)$ because ξ_i contains an e_j with $1 \leq j \leq n - k, Df(e_j) = 0$ on $C \cap \{f = \varphi(c)\}$, and Df is Lipschitz. Now by the general formula for a third derivative (6) and (1) we have

$$\begin{aligned} D^2(J_k f)^2 =& O(1) + 0 + O(r^{-2}) D(p_2 - p_1)(\xi) \\ &+ O(r^{-1}) DD(p_2 - p_1)(\xi) + O(1) \\ =& O(1) \end{aligned}$$

by (3). Therefore the weighting factor $J_k f$ and hence the whole metric is $C^{1,1}$.

Conclusion. For any competitor S' with the same boundary as S, almost every slice $S'_c = S' \cap \{f = \varphi(c)\}$ has the same boundary as S_c, by general slicing theory [5, 4.3.1, 4.3.8]. Because each slice S_c consists of three geodesics meeting at 120-degree angles in a $C^{1,1}$ metric, by Proposition 4.1, $\text{length}(S_c) \leq \text{length}(S'_c)$. Therefore

$$\text{area}(S) = \int \text{length}(S_c) \leq \int \text{length}(S'_c) \leq \text{area}(S')$$

by the coarea formula [14, 3.13]. (The final inequality need not be equality since S' need not be orthogonal to the slicing sets $\{f = \varphi(c)\}$.) Hence we conclude that S is area minimizing.

5.2. *Area minimization at boundary.* There are 10 conjectured types of smooth boundary singularities of soap films in $\mathbf{R}^3$, as pictured in Figure 5.3 (see [17, 11.3], [19]):

(1) a smooth minimal submanifold with boundary,

(2) a minimal submanifold that just happens to contain a given smooth boundary curve,

(3) a triple junction that just happens to contain a given smooth boundary curve tangent to the singular curve at the point,

(4) a portion of a triple junction with given smooth boundary in one surface tangent to (or partly contained in) the singular curve, the outside portion of that surface discarded,

(5) two minimal surfaces meeting smoothly at an angle $\alpha \geq 120^o$,

(6) a smooth transition from (2) to (5), called *creasing*, as occurs in a portion of a neighborhood of a classical branch point with one self-intersection curve as boundary,

(7) a smooth transition from (5) to (4) (type (4) includes trivial examples with $\alpha = 120^o$),

(8) a generalization of (3) allowing creasing (6),

(9) a smooth tetrahedral singularity which just happens to contain a given smooth boundary curve,

(10) a generalization of (9) allowing creasing.

It follows immediately from Theorems 2.1 and 5.1 that types (2), (3), and (4) are locally area minimizing. The methods likewise show that types (1), (5), (6), and (7) are locally minimizing (where smooth means C^4). Of course (9) would follow from the corresponding result without boundary, currently known only for the flat tetrahedral cone itself. Types (8) and (10) remain open; it is easy to imagine that if the creasing is severe, the surface need not be locally minimizing. No examples of types (8) or (10) are known.

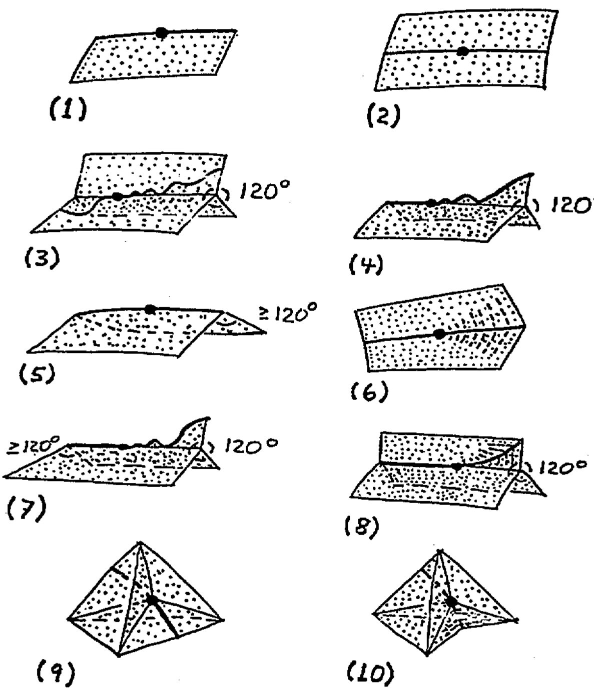

FIGURE 5.3. Of the 10 conjectured types of boundary singularities, at least (1)-(7) are locally area minimizing.

As in 5.1, "area minimizing" means size minimizing for an appropriate assignment of orientations and multiplicities. For cases (2), (5), and (6), the surfaces are oriented so that the boundary has multiplicity 2. For cases (3) and (8), the three surfaces are oriented so that the boundary has multiplicity 3; when the boundary diverts along one surface, the inside portion has the opposite orientation with multiplicity 2. Cases (4) and (7) are similar, without the outside piece of surface; the boundary has multiplicity 2. For cases (9) and (10), the surfaces can be oriented with multiplicity 1 or 2 so that the boundary has multiplicity 2.

More generally consider singularities where the boundary curve leaves the surface, as in generalizations of cases (2), (3), (6), (8), (9), and (10), as pictured in Figure 5.4. Cases (2'), (3'), and (9'), in which the given boundary is incidental, remain locally area minimizing. For cases (6'), (8'), and (10'), there is of course no way to orient a surface to give it just part of a curve as boundary. One requires another setting, such as Brakke's covering spaces [2]. Some such singularity, probably usually (6'), must occur in Brakke's minimizers, but no particular minimizing example is known. Case (6'), which as (6) arises in a classical branch point, was analyzed in some detail by Brakke [1]. If the boundary curve leaves the surface on the side of the exterior of the creasing angle, then the surface is probably not locally area minimizing in any sense; any setting which allowed the curve to so separate would presumably allow more of the surface to pull of the wire in a favorable direction.

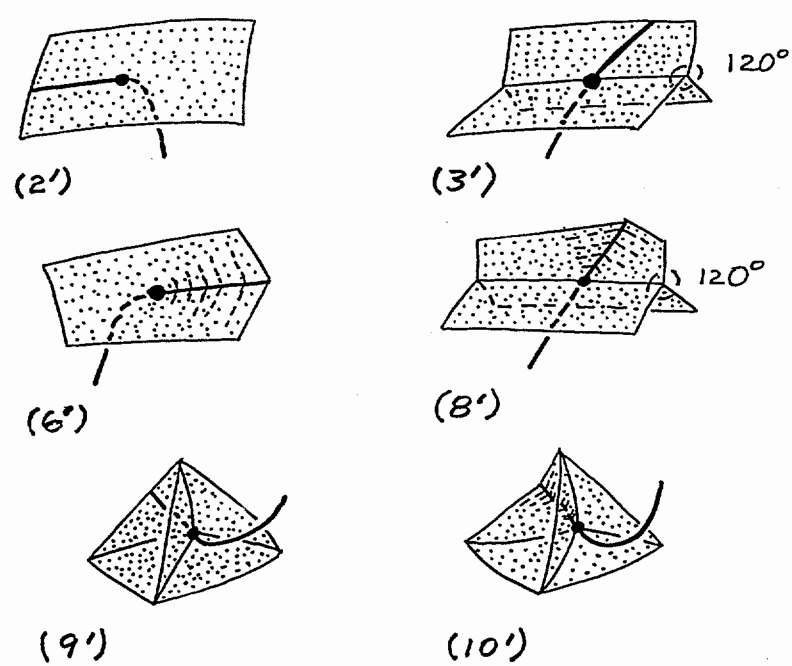

FIGURE 5.4. Of these 6 conjectured types of generalized boundary singularities, where the boundary curve leaves the surface, only the trivial ones (2'), (3'), (9') are known to be locally area minimizing, although some nontrivial type must occur in Brakke's minimizers.

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